Invariant Measure for Linear Stochastic PDEs in the Space of Tempered Distributions

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To avoid notational complexity, we discuss results in dimension one.

Description

- In this talk , we discuss about the existence and uniqueness of Invariant measure for linear Stochastic PDEs with potential in the space of tempered distributions using Monotonicity inequality.
- Work on Problem “Invariant Measure for Linear Stochastic PDEs in the Space of Tempered Distributions” is available on arxiv “https://arxiv.org/abs/2405.19926”

Outline

1. Preliminaries
   - Linear Stochastic PDE with potential
   - Tempered Distributions, Hermite-Sobolev Spaces and Some Operators
2. Known results
   - Monotonicity Inequality
   - Existence and Uniqueness of Stochastic PDEs
3. Invariant measure
   - New Result

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A Linear Stochastic PDE with potential

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Consider the Stochastic PDE:

$$dX_t = (L^* - \alpha)(X_t)\, dt + A^*(X_t)\, dB_t; \quad X_0 = \phi \tag{1.1}$$

for some $p, \alpha \in \mathbb{R}$, where

1. A^* and L^* are linear Differential operators involving multiplication by smooth functions σ, b and distributional derivatives ∂
2. X_t takes value in $\mathcal{S}'$
3. $\{B_t\}_{t \geq 0}$ is a Brownian motion
4. ϕ is $\mathcal{F}_0$- measurable, $\mathcal{S}_p$- valued random variable

In the literature, existence and uniqueness of strong solution has already been investigated. We are interested in the long term behaviour of the solution, more specifically about invariant measures.

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Tempered Distributions

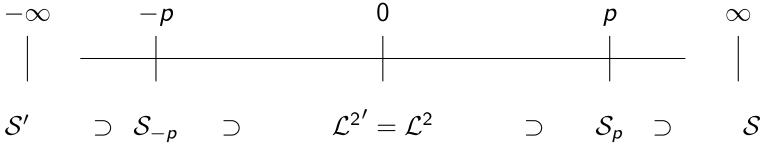
- Let $\mathcal{S}$ denote the space of real valued rapidly decreasing smooth functions on $\mathbb{R}$ (Schwartz class) with dual $\mathcal{S}'$, the space of tempered distributions.
- For $p \in \mathbb{R}$, consider the increasing norms $\|\cdot\|_p$, defined by the inner products

$$\langle f, g\rangle_p := \sum_{k=0}^{\infty}(2k+1)^{2p}\langle f, h_k\rangle\langle g, h_k\rangle, \quad f, g \in \mathcal{S}.$$

Here, $\{h_k\}_{k=0}^{\infty}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}, dx)$ given by Hermite functions $h_k(t) := (2^k k!\sqrt{\pi})^{-1/2}\exp\{-t^2/2\}H_k(t), t \in \mathbb{R}$, where H_k are the Hermite polynomials.

Hermite-Sobolev spaces

The Hermite-Sobolev spaces[1] $\mathcal{S}_p, p \in \mathbb{R}$ are defined to be the completion of $\mathcal{S}$ in $\|\cdot\|_p$. It can be shown that $(\mathcal{S}_{-p}, \|\cdot\|_{-p})$ is isometrically isomorphic to the dual of $(\mathcal{S}_p, \|\cdot\|_p)$ for $p \geq 0$.



[1] Kiyosi Itô, *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

Derivative operator

- Given a tempered distribution $\psi \in \mathcal{S}'$, the distributional derivative of ψ is defined via the following relation

$$\langle \partial\psi \, , \, \phi \rangle := - \langle \psi \, , \, \partial\phi \rangle \, , \, \forall \phi \in \mathcal{S}.$$

- $\partial : \mathcal{S}_p \to \mathcal{S}_{p-\frac{1}{2}}$ is a bounded linear operator. So the Laplacian $\triangle = \partial^2$ is a bounded linear operator from $\mathcal{S}_p$ to $\mathcal{S}_{p-1}$.

Multiplication operator

Given a smooth function σ, we define the multiplication by σ on $\mathcal{S}'$ by

$$\langle \sigma\psi \, , \, \phi \rangle := \langle \psi \, , \, \sigma\phi \rangle \, , \, \forall \phi \in \mathcal{S}$$

and then extend via duality.

Outline

Definition of Monotonicity Inequality

Definition

We say that a pair of linear operators (L^*, A^*) satisfy the Monotonicity inequality in $\|\cdot\|_p$, if we have

$$2\,\langle \phi\,,\,L^*\phi\rangle_p + \|A^*\phi\|_p^2 \le C\|\phi\|_p^2, \quad \forall \phi \in \mathcal{S}. \tag{2.1}$$

Remark (Importance of Monotonicity inequality)

1. *The Monotonicity inequality was introduced in [Krylov and Rozovskii, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow 1979]. This describes a sufficient condition for uniqueness of solution to SPDE's.*
2. *In [L. Gawarecki, V. Mandrekar, and B. Rajeev - Theory of Stochastic Processes 2008], it was shown that the monotonicity inequality can be used to prove both the existence and the uniqueness of solution to Linear SPDE's.*

Known results

Consider the following operators,

$$A^*\psi := -\partial(\sigma\psi), \quad L^*\psi := \frac{1}{2}\partial^2(\sigma^2\psi) - \partial(b\psi)$$

The Monotonicity inequality (for the case involving linear A^*, L^*) has been explicitly proved for the following cases.

- σ, b are constants, $p \in \mathbb{R}$ (Gawarecki, Mandrekar and Rajeev - Infinite Dimensional Analysis, Quantum Probability and Related Topics 2009)
- $b(x) = \alpha + \beta x, \forall x$ (for fixed α, β), σ is a constant and $p \in \mathbb{R}$. [Bhar and Rajeev - Proceedings of Indian Academy of Sciences (Mathematical Sciences) 2015]
- σ, b are bounded smooth functions on $\mathbb{R}$ with bounded derivatives and $p = 0$ ($\mathcal{S}_0 = \mathcal{L}^2(\mathbb{R})$) [Bhar, Bhaskaran and Sarkar - Potential Analysis 2020]

Outline

(An Application of [4, Theorem 1])[2]

Theorem

Assume that $\mathbb{E}\|\phi\|_p^2 < \infty$ *and the pair of linear operators* (L^*, A^*) *satisfy the Monotonicity inequality. Then the Linear Stochastic PDE (1.1) has an* $\mathcal{S}_p$*–valued strong solution* $\{X_t : t \geq 0\}$ *in* $\mathcal{S}_p$ *satisfying the integral equation*

$$X_t = \phi + \int_0^t (L^* - \alpha)(X_s)\, ds + \int_0^t A^*(X_s)\, dB_s \tag{2.2}$$

in $\mathcal{S}_{p-2}$ *and the solution is unique.*

[2] Gawarecki, L. and Mandrekar, V. and Rajeev, B., *Foundations of stochastic differential equations in infinite-dimensional spaces*, Theory Stoch. Process. 14 (2008), no. 2, 28–34. MR 2479730

Outline

New Result

Lemma

The inclusion $i : \mathcal{S}_p \to \mathcal{S}_q$ is compact if $q < p$.

Definition

We say that a probability measure μ on $\mathcal{S}_p$ (for some $p \geq 0$) is invariant for a time-homogeneous Markov process $\{X_t^x, t \geq 0\}$ with the related Feller semigoup $\{P_t, t \geq 0\}$ if for all $A \in \mathcal{B}(\mathcal{S}_p)$, $\mu(A) = \int_{\mathcal{S}_p} P(t, y, A)\mu(dy)$ or equivalently, since $\mathcal{S}_p$ is a Polish space, if for $f \in C_b(\mathcal{S}_p)$,

$$\int_{\mathcal{S}_p} (P_t f) d\mu = \int_{\mathcal{S}_P} f d\mu$$

Theorem

Let $2\alpha > C$, $i : \mathcal{S}_{p-2} \hookrightarrow \mathcal{S}_{q-2}$ for $p > q$ be a compact continuous embedding and the Monotonicity inequality holds for $p, p-2, q-2$ then there exists unique invariant measure μ for the SPDE (1.1) in $(\mathcal{S}_{q-2}, \|.\|_{q-2})$.

Idea of proof

Let $\{X_t^x, t \geq 0\}$ be the $\mathcal{S}_p$- valued solution of SPDE (1.1), we have

$$X_t^x = x + \int_0^t (L^* - \alpha)(X_s^x) ds + \int_0^t A^*(X_s^x)\, dB_s$$

and the equality holds in $\mathcal{S}_{p-2}$. By applying Itô rule, Monotonicity inequality and taking expectation,

$$\begin{aligned}
\mathbb{E}\, \|X_t^x\|_{p-2}^2 &= \|x\|_{p-2}^2 + \mathbb{E} \int_0^t \left(2 \langle X_s^x , (L^* - \alpha)(X_s^x) \rangle_{p-2} + \|A^* X_s^x\|_{p-2}^2 \right) ds \\
&= \|x\|_{p-2}^2 + \int_0^t \mathbb{E} \left(2 \langle X_s^x , (L^* - \alpha)(X_s^x) \rangle_{p-2} + \|A^* X_s^x\|_{p-2}^2 \right) ds \\
&\leq \|x\|_{p-2}^2 + (C - 2\alpha) \int_0^t \mathbb{E}\, \|X_s^x\|_{p-2}^2\, ds
\end{aligned}$$

By Gronwall's inequality, we have

$$\mathbb{E}\, \|X_t^x\|_{p-2}^2 \leq \|x\|_{p-2}^2\, e^{-(2\alpha - C)t}. \tag{3.1}$$

For any $R > 0$, we have

$$P(\|X_t^x\|_{p-2} \geq R) \leq \frac{1}{R^2}\mathbb{E}\|X_t^x\|_{p-2}^2,$$

and hence by (3.1) , we have

$$\lim_{R\to\infty} \sup_T \frac{1}{T}\int_0^T P(\|X_t^x\|_{p-2} \geq R)dt = 0.$$

The embedding $\mathcal{S}_{p-2} \hookrightarrow \mathcal{S}_{q-2}$ is compact for $q < p$, and hence the set $\{x \in \mathcal{S}_{p-2} : \|x\|_{p-2} \leq R_\epsilon\}$ is compact in $\mathcal{S}_{q-2}$. There exists an invariant measure μ on $\mathcal{S}_{q-2}$ for SPDE (1.1) by Prokhorov's theorem [6, Corollary 7.4] and μ is unique using arguments similar to [6, Theorem 7.13].

References I

- Suprio Bhar, *Characterizing Gaussian flows arising from Itô's stochastic differential equations*, Potential Anal. **46** (2017), no. 2, 261–277. MR 3647575
- Suprio Bhar, Rajeev Bhaskaran, and Barun Sarkar, *Stochastic PDEs in $\mathcal{S}'$ for SDEs driven by Lévy noise*, Random Oper. Stoch. Equ. **28** (2020), no. 3, 217–226. MR 4148176
- Suprio Bhar and B. Rajeev, *Differential operators on Hermite Sobolev spaces*, Proc. Indian Acad. Sci. Math. Sci. **125** (2015), no. 1, 113–125. MR 3331916
- L. Gawarecki, V. Mandrekar, and B. Rajeev, *Linear stochastic differential equations in the dual of a multi-Hilbertian space*, Theory Stoch. Process. **14** (2008), no. 2, 28–34. MR 2479730
- ______, *The monotonicity inequality for linear stochastic partial differential equations*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), no. 4, 575–591. MR 2590157

References II

- Leszek Gawarecki and Vidyadhar Mandrekar, *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR 2560625
- Xuan Da Hu, *Boundedness and invariant measures of semilinear stochastic evolution equations*, Nanjing Daxue Xuebao Shuxue Bannian Kan **4** (1987), no. 1, 1–14. MR 916950
- Akira Ichikawa, *Semilinear stochastic evolution equations: boundedness, stability and invariant measures*, Stochastics **12** (1984), no. 1, 1–39. MR 738933
- Kiyosi Itô, *Foundations of stochastic differential equations in infinite-dimensional spaces*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 47, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984. MR 771478

References III

- Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR 1121940
- Achim Klenke, *Probability theory*, second ed., Universitext, Springer, London, 2014, A comprehensive course. MR 3112259
- N. V. Krylov and B. L. Rozovskiĭ, *Stochastic evolution equations*, Current problems in mathematics, Vol. 14 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147, 256. MR 570795
- Peter Mörters and Yuval Peres, *Brownian motion*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 30, Cambridge University Press, Cambridge, 2010, With an appendix by Oded Schramm and Wendelin Werner. MR 2604525
- Bernt Ø ksendal, *Stochastic differential equations*, fifth ed., Universitext, Springer-Verlag, Berlin, 1998, An introduction with applications. MR 1619188

References IV

B. Rajeev and S. Thangavelu, *Probabilistic representations of solutions to the heat equation*, Proc. Indian Acad. Sci. Math. Sci. **113** (2003), no. 3, 321–332. MR 1999259

______, *Probabilistic representations of solutions of the forward equations*, Potential Anal. **28** (2008), no. 2, 139–162. MR 2373102

Thank You